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LETTER TO THE EDITOR

A class of generalised multiple hypergeometric series arising in physical and quantum chemical applications

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Abstract. The multivariable hypergeometric function ${}^nF(x_1, \ldots, x_n)$, considered recently by Niukkanen, is a straightforward generalisation of certain well known hypergeometric functions of n variables; indeed it provides a unification of the generalised hypergeometric function ${}_pF_q$ of one variable, Appell and Kampé de Fériet functions of two variables, and Lauricella functions of n variables, as well as of many other hypergeometric series which arise naturally in physical and quantum chemical applications. The object of the present paper is to derive several interesting properties of this multivariable hypergeometric function (including, for example, many which were not given by Niukkanen) as useful consequences of substantially more general results available in the literature.

Hypergeometric series in one and more variables occur frequently in a wide variety of problems in theoretical physics, applied mathematics, engineering sciences, statistics, and operations research (see, e.g., Exton 1976, ch 7, 8, 1978 ch 7, Carlson 1977, Srivastava and Kashyap 1982, Kabe 1962, Srivastava and Exton 1979, Dyer 1982). Motivated by a vast field of physical and quantum chemical applications of these hypergeometric functions, Niukkanen (1983, 1984) presented various interesting and useful properties of a generalised hypergeometric series of n variables defined by

$$F^{p_0:\, p_1;\, \ldots;\, p_n}_{q_0:\, q_1;\, \ldots;\, q_n}\left(\begin{matrix}\boldsymbol{a}_0:\, \boldsymbol{a}_1;\, \ldots;\, \boldsymbol{a}_n;\\ \boldsymbol{b}_0:\, \boldsymbol{b}_1;\, \ldots;\, \boldsymbol{b}_n;\end{matrix}\; x_1, \ldots, x_n\right) = \sum_{s_1, \ldots, s_n = 0}^{\infty} \frac{(\boldsymbol{a}_0)_{s_1+\ldots+s_n}}{(\boldsymbol{b}_0)_{s_1+\ldots+s_n}} \prod_{j=1}^{n} \frac{(\boldsymbol{a}_j)_{s_j}}{(\boldsymbol{b}_j)_{s_j}} \frac{x_j^{s_j}}{s_j!}, \tag{1}$$

where, for convenience,

$$\boldsymbol{a}_j = (a_j^1, \ldots, a_j^{p_j}), \qquad \boldsymbol{b}_j = (b_j^1, \ldots, b_j^{q_j}), \tag{2}$$

$$(\boldsymbol{a}_j)_s = \prod_{k=1}^{p_j} (a_j^k)_s, \qquad (\boldsymbol{b}_j)_s = \prod_{k=1}^{q_j} (b_j^k)_s, \tag{3}$$

and $(\lambda)_s = \Gamma(\lambda + s)/\Gamma(\lambda)$ is a Pochhammer symbol. Thus $\boldsymbol{a}_j$ and $\boldsymbol{b}_j$ $(j = 0, 1, \ldots, n)$ are vectors with dimensions p_j and q_j, respectively.

The multivariable hypergeometric function (1) is an obvious special case of the generalised Lauricella function of n variables, which was first introduced and studied by Srivastava and Daoust (1969, p 454 *et seq.*). In fact, this widely studied (Srivastava-Daoust) generalised Lauricella function has appeared in several subsequent works including, for example, two important books by Exton (1976, § 3.7, 1978, § 1.4) and

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a recent book by Srivastava and Manocha (1984, p 64 *et seq.*). Also, a further special case of the multivariable hypergeometric function (1) when

$$p_1 = \ldots = p_n \qquad \text{and} \qquad q_1 = \ldots = q_n \tag{4}$$

was considered earlier by Karlsson (1973). In the present letter we aim at employing these fruitful connections of (1) with much more general multiple hypergeometric functions (studied in the literature rather systematically) in order to derive several interesting properties of (1). Many of the results presented here were not given by Niukkanen (1983, 1984).

Following Srivastava and Daoust (1969, p 454 *et seq*) we find it to be convenient to abbreviate the left-hand side of (1) simply by

$$F^{p_0:\, p_1;\, \ldots;\, p_n}_{q_0:\, q_1;\, \ldots;\, q_n}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \tag{5}$$

whenever no confusion is likely to arise.

Niukkanen (1983, 1984) presented the definition (1) without stating the regions of convergence of the multiple hypergeometric series occurring on the right-hand side of (1). However, from the work of Srivastava and Daoust (1972, § 5), where the regions of convergence of the (Srivastava-Daoust) generalised Lauricella series in n variables are given, we readily observe that the multiple hypergeometric series in (1) converges absolutely when

$$1 + q_0 + q_k - p_0 - p_k \geqslant 0, \qquad k = 1, \ldots, n, \tag{6}$$

where the equality holds true provided, in addition, we have either

$$p_0 > q_0 \qquad \text{and} \qquad |x_1|^{1/(p_0-q_0)} + \ldots + |x_n|^{1/(p_0-q_0)} < 1 \tag{7}$$

or

$$p_0 \leqslant q_0 \qquad \text{and} \qquad \max\{|x_1|, \ldots, |x_n|\} < 1. \tag{8}$$

Indeed, under certain parametric constraints, the multiple hypergeometric series in (1) converges also when

By analogy with the abbreviations in (2), let

$$\boldsymbol{a} = (a^1, \ldots, a^p) \qquad \text{and} \qquad \boldsymbol{b} = (b^1, \ldots, b^q), \tag{10}$$

so that $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors with dimensions p and q, respectively. The following trivial reduction formulae for the multivariable hypergeometric function (1) are given by Niukkanen (1983)†:

$$F^{0:\, p_1;\, \ldots;\, p_n}_{0:\, q_1;\, \ldots;\, q_n}\left(\begin{matrix} -:\, \boldsymbol{a}_1;\, \ldots;\, \boldsymbol{a}_n; \\ -:\, \boldsymbol{b}_1;\, \ldots;\, \boldsymbol{b}_n; \end{matrix}\; x_1, \ldots, x_n\right) = {}_{p_1}F_{q_1}\left[\begin{matrix} \boldsymbol{a}_1; \\ \boldsymbol{b}_1; \end{matrix}\; x_1\right] \ldots {}_{p_n}F_{q_n}\left[\begin{matrix} \boldsymbol{a}_n; \\ \boldsymbol{b}_n; \end{matrix}\; x_n\right], \tag{11}$$

which follows immediately from the definition (1);

$$F^{p:\, 0;\, \ldots;\, 0}_{q:\, 0;\, \ldots;\, 0}\left(\begin{matrix} \boldsymbol{a}:\, -;\, \ldots;\, -; \\ \boldsymbol{b}:\, -;\, \ldots;\, -; \end{matrix}\; x_1, \ldots, x_n\right) = {}_pF_q\left[\begin{matrix} \boldsymbol{a}; \\ \boldsymbol{b}; \end{matrix}\; x_1 + \ldots + x_n\right], \tag{12}$$

which is an obvious special case of the elementary multinomial theorem (see, e.g.,

† Throughout the present paper we follow the standard notation ${}_pF_q$ to denote a generalised hypergeometric series in one variable with p numerator and q denominator parameters. Also, as is quite usual in the theory of hypergeometric functions, an empty set of parameters is represented by a dash.

Srivastava (1971, p 4, equation (12))

$$\sum_{s_1,\ldots,s_n=0}^{\infty} f(s_1+\ldots+s_n)\frac{x_1^{s_1}}{s_1!}\cdots\frac{x_n^{s_n}}{s_n!}=\sum_{s=0}^{\infty} f(s)\frac{(x_1+\ldots+x_n)^s}{s!}. \tag{13}$$

From Panda (1974) and Karlsson (1982, 1983), involving much more general considerations, we also have the following non-trivial reduction formulae for the multivariable hypergeometric function (1):

$$F^{p:\,1;\,\ldots;\,1}_{q:\,0;\,\ldots;\,0}\left(\begin{matrix}\mathbf{a}: & \nu_1; & \ldots; & \nu_n; \\ \mathbf{b}: & -; & \ldots; & -;\end{matrix}\; x,\ldots,x\right) = {}_{p+1}F_q\left[\begin{matrix}\mathbf{a},\ \nu_1+\ldots+\nu_n; \\ \mathbf{b};\end{matrix}\; x\right], \tag{14}$$

$$F^{p:\,2;\,\ldots;\,2}_{q:\,1;\,\ldots;\,1}\left(\begin{matrix}\mathbf{a}: & \nu_1,\ \nu_1+1/2; & \ldots; & \nu_n,\ \nu_n+1/2; \\ \mathbf{b}: & 2\nu_1+1; & \ldots; & 2\nu_n+1;\end{matrix}\; x,\ldots,x\right) = {}_{p+2}F_{q+1}\left[\begin{matrix}\mathbf{a},\ \nu_1+\ldots+\nu_n,\ \nu_1+\ldots+\nu_n+1/2; \\ \mathbf{b},\ 2\nu_1+\ldots+2\nu_n+1;\end{matrix}\; x\right], \tag{15}$$

$$F^{p:\,2;\,\ldots;\,2}_{q:\,1;\,\ldots;\,1}\left(\begin{matrix}\mathbf{a}: & \nu_1,\ \omega\nu_1+1; & \ldots; & \nu_n,\ \omega\nu_n+1; \\ \mathbf{b}: & \omega\nu_1; & \ldots; & \omega\nu_n;\end{matrix}\; x,\ldots,x\right) = F^{p:\,1;\,1}_{q:\,0;\,0}\left[\begin{matrix}\mathbf{a}: & \nu_1+\ldots+\nu_n+n; & -n; \\ \mathbf{b}: & -; & -;\end{matrix}\; x,(1-\omega^{-1})x\right], \tag{16}$$

where the function on the right-hand side is a two-variable case of (1).

For $\omega=1$, (16) evidently leads to (14). A special case of (16) when $\omega=\frac{1}{2}$ yields the interesting result (cf Karlsson 1983)

$$F^{p:\,2;\,\ldots;\,2}_{q:\,1;\,\ldots;\,1}\left(\begin{matrix}\mathbf{a}: & \nu_1,\ \nu_1/2+1; & \ldots; & \nu_n,\ \nu_n/2+1; \\ \mathbf{b}: & \nu_1/2; & \ldots; & \nu_n/2;\end{matrix}\; x,\ldots,x\right) = {}_{2p+1}F_{2q}\left[\begin{matrix}\mathbf{a}/2,\ \mathbf{a}/2+1/2,\ -n; \\ \mathbf{b}/2,\ \mathbf{b}/2+1/2;\end{matrix}\; 4^{p-q}x^2\right], \tag{17}$$

where $\nu_1+\ldots+\nu_n=-2n$.

For suitable choices of the variable x and of the various parameters involved, the one-variable hypergeometric series occurring on the right-hand sides of the above reduction formulae can be summed by appealing to one or the other known summation theorems (see Slater 1966, Luke 1975). For example, in view of Dougall's summation theorem, the reduction formula (14) immediately yields (cf Srivastava 1978)

$$\begin{aligned}&F^{6:\,1;\,\ldots;\,1}_{6:\,0;\,\ldots;\,0}\left(\begin{matrix}\alpha,\ 1+\alpha/2, & \beta, & \gamma, & \delta, & -N: & \nu_1; & \ldots; & \nu_n; \\ \alpha/2,\ 1+\alpha-\beta, & 1+\alpha-\gamma, & 1+\alpha-\delta, & 1+\alpha+N, & 1+\alpha-\Delta: & -; & \ldots; & -;\end{matrix}\; 1,\ldots,1\right)\\ &\qquad=\frac{(1+\alpha)_N(1+\alpha-\beta-\gamma)_N(1+\alpha-\beta-\delta)_N(1+\alpha-\gamma-\delta)_N}{(1+\alpha-\beta)_N(1+\alpha-\gamma)_N(1+\alpha-\delta)_N(1+\alpha-\beta-\gamma-\delta)_N},\end{aligned} \tag{18}$$

where $N\in\{0,1,2,\ldots\}$, and

$$1+2\alpha=\beta+\gamma+\delta+\Delta-N, \qquad \Delta\equiv\nu_1+\ldots+\nu_n.$$

Similarly, by employing Bailey's summation theorem in the reduction formula (15) we obtain

$$\begin{aligned}&F^{2:\,2;\,\ldots;\,2}_{2:\,1;\,\ldots;\,1}\left(\begin{matrix}-N, & \lambda+N: & \mu_1,\ \mu_1+1/2; & \ldots; & \mu_n,\ \mu_n+1/2; \\ \lambda/2, & \lambda/2+1/2: & 2\mu_1+1; & \ldots; & 2\mu_n+1;\end{matrix}\; 1,\ldots,1\right)\\ &\qquad=\frac{(\lambda-2\mu_1-\ldots-2\mu_n)_N}{(\lambda)_N}, \qquad \forall N\in\{0,1,2,\ldots\},\end{aligned} \tag{19}$$

which incidentally was derived by Srivastava (1977, p 452, equation (18)) from the reduction formula (14).

The earlier works of Srivastava and Daoust (1969), Srivastava and Panda (1973, 1974, 1975, 1976), and Srivastava (1981) contain several general classes of analytic or asymptotic expansions and multiplication formulae for the (Srivastava–Daoust) generalised Lauricella function of n variables. Each of these general results can indeed be applied in order to derive the corresponding expansion and multiplication theorems for the multivariable hypergeometric function (1). We present here only the following three classes of expansion or multiplication formulae:

$$F(ux_1, \ldots, ux_n) \equiv F^{p+p_0:\, p_1;\, \ldots;\, p_n}_{q+q_0:\, q_1;\, \ldots;\, q_n}\left(\begin{matrix}\boldsymbol{a},\ \boldsymbol{a}_0:\ \boldsymbol{a}_1;\ \ldots;\ \boldsymbol{a}_n;\\ \boldsymbol{b},\ \boldsymbol{b}_0:\ \boldsymbol{b}_1;\ \ldots;\ \boldsymbol{b}_n;\end{matrix}\ ux_1, \ldots, ux_n\right)$$

$$= \sum_{m=0}^{\infty} \frac{\Gamma_m(\boldsymbol{a}, \boldsymbol{c}; \boldsymbol{b}, \boldsymbol{d})}{(\lambda+m)_m} \frac{(-u)^m}{m!}\, {}_{p+r}F_{q+s+1}\left[\begin{matrix} & \boldsymbol{a}+m,\ \boldsymbol{c}+m;\\ \lambda+2m+1, & \boldsymbol{b}+m,\ \boldsymbol{d}+m;\end{matrix}\ u\right]$$

$$\times F^{2+s+p_0:\, p_1;\, \ldots;\, p_n}_{r+q_0:\, q_1;\, \ldots;\, q_n}\left(\begin{matrix}-m,\ \lambda+m,\ \boldsymbol{d},\ \boldsymbol{a}_0:\ \boldsymbol{a}_1;\ \ldots;\ \boldsymbol{a}_n;\\ \boldsymbol{c},\ \boldsymbol{b}_0:\ \boldsymbol{b}_1;\ \ldots;\ \boldsymbol{b}_n;\end{matrix}\ x_1, \ldots, x_n\right), \tag{20}$$

$$p+r \leqslant q+s+2 \qquad \text{(the equality holds true when } |u|<1\text{)};$$

$$F(ux_1, \ldots, ux_n) = \sum_{m=0}^{\infty} \Gamma_m(\boldsymbol{a}, \boldsymbol{c}; \boldsymbol{b}, \boldsymbol{d}) \frac{(-u)^m}{m!}\, {}_{p+r}F_{q+s}\left[\begin{matrix}\boldsymbol{a}+m,\ \boldsymbol{c}+m;\\ \boldsymbol{b}+m,\ \boldsymbol{d}+m;\end{matrix}\ u\right]$$

$$\times F^{1+s+p_0:\, p_1;\, \ldots;\, p_n}_{r+q_0:\, q_1;\, \ldots;\, q_n}\left(\begin{matrix}-m,\ \boldsymbol{d},\ \boldsymbol{a}_0:\ \boldsymbol{a}_1;\ \ldots;\ \boldsymbol{a}_n;\\ \boldsymbol{c},\ \boldsymbol{b}_0:\ \boldsymbol{b}_1;\ \ldots;\ \boldsymbol{b}_n;\end{matrix}\ x_1, \ldots, x_n\right), \tag{21}$$

$$p+r \leqslant q+s+1 \qquad \text{(the equality holds true when } |u|<1\text{)};$$

$$F(ux_1, \ldots, ux_n) = \beta \sum_{m=0}^{\infty} (1-\alpha m+\beta)_{m-1} \Gamma_m(\boldsymbol{a}, \boldsymbol{c}; \boldsymbol{b}, \boldsymbol{d}) \frac{(-u)^m}{m!}$$

$$\times\, {}_{p+r+1}F_{q+s}\left[\begin{matrix}(1-\alpha)m+\beta,\ \boldsymbol{a}+m,\ \boldsymbol{c}+m;\\ \boldsymbol{b}+m,\ \boldsymbol{d}+m;\end{matrix}\ u\right]$$

$$\times F^{2+s+p_0:\, p_1;\, \ldots;\, p_n}_{2+r+q_0:\, q_1;\, \ldots;\, q_n}\left(\begin{matrix} & -m,\ 1+\beta/(1-\alpha),\ \boldsymbol{d}_0,\ \alpha_0:\ \boldsymbol{a}_1;\ \ldots;\ \boldsymbol{a}_n;\\ \beta/(1-\alpha),\ 1-\alpha m+\beta, & \boldsymbol{c},\ \boldsymbol{b}_0:\ \boldsymbol{b}_1;\ \ldots;\ \boldsymbol{b}_n;\end{matrix}\ x_1, \ldots, x_n\right), \qquad \beta \neq 0, \tag{22}$$

$$p+r \leqslant q+s \qquad \text{(the equality holds true when } |u|<1\text{)};$$

where, by analogy with the abbreviations in (2) and (10),

$$\boldsymbol{c} = (c^1, \ldots, c^r) \qquad \text{and} \qquad \boldsymbol{d} = (d^1, \ldots, d^s), \tag{23}$$

so that $\boldsymbol{c}$ and $\boldsymbol{d}$ are vectors with dimensions r and s, respectively, and

$$\Gamma_m(\boldsymbol{a}, \boldsymbol{c}; \boldsymbol{b}, \boldsymbol{d}) = \frac{(\boldsymbol{a})_m(\boldsymbol{c})_m}{(\boldsymbol{b})_m(\boldsymbol{d})_m}, \qquad m \geqslant 0, \tag{24}$$

it being understood in every case that

$$1+q_0+q_k-p_0-p_k \geqslant p-q, \qquad k=1, \ldots, n, \tag{25}$$

where the equality holds true when $|u|$ and $|x_1|, \ldots, |x_n|$ are appropriately constrained in accordance with (7) and (8).

The expansion formula (20) follows readily from a more general result due to Srivastava and Daoust (1969, p 456, equation (4.3)). In view of the principle of confluence exhibited by

$$\lim_{\lambda\to\infty}\left\{(\lambda)_m\left(\frac{z}{\lambda}\right)^m\right\} = z^m = \lim_{\mu\to\infty}\left\{\frac{(\mu z)^m}{(\mu)_m}\right\}, \tag{26}$$

for bounded z and $m = 0, 1, 2, \ldots$, the expansion formula (21) will follow if in (20) we replace u by λu and x_k by x_k/λ, $k = 1, \ldots, n$, and let $\lambda \to \infty$. In a much more general context, (21) was given by Srivastava and Panda (1976, p 143, equation (6.6)). The expansion formula (22), which also yields (21) in the special case $\alpha = 0$, follows from a general expansion due to Srivastava (1981, p 302, equation (3.3)).

Each of the above expansion or multiplication formulae (20), (21) and (22) can be further specialised to yield a desired Neumann expansion (in series of Bessel functions J_ν and I_ν), since

$$J_\nu(z)=\frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left[\begin{matrix}\overline{};\\ \nu+1;\end{matrix}\,-z^2/4\right]=\frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}\,\mathrm{e}^{\pm\mathrm{i}z}\,{}_1F_1\left[\begin{matrix}\nu+1/2;\\ 2\nu+1;\end{matrix}\,\mp2\mathrm{i}z\right] \tag{27}$$

and

$$I_\nu(z)=\frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left[\begin{matrix}\overline{};\\ \nu+1;\end{matrix}\,z^2/4\right]=\frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}\,\mathrm{e}^{\pm z}\,{}_1F_1\left[\begin{matrix}\nu+1/2;\\ 2\nu+1;\end{matrix}\,\mp2z\right]. \tag{28}$$

We should remark in passing that a very specialised version of the expansion formula (20) was given by Niukkanen (1983, p 1823, equation (47)).

Linear, bilinear and bilateral generating functions involving various general classes of functions of n variables have been studied rather widely in the literature. Much of the important work on the subject has been presented systematically in the recent monograph on generating functions by Srivastava and Manocha (1984, ch 4, 7, 8, 9). We choose to give here only the following simple consequences of certain general classes of linear generating functions due to Srivastava (1970, 1972):

$$\sum_{m=0}^{\infty}\binom{\alpha+(\beta+1)m}{m}F^{1+p_0:\,p_1;\,\ldots;\,p_n}_{1+q_0:\,q_1;\,\ldots;\,q_n}\left(\begin{matrix}-m,\ \boldsymbol{a}_0:\ \boldsymbol{a}_1;\ \ldots;\ \boldsymbol{a}_n;\\ \alpha+\beta m+1,\ \boldsymbol{b}_0:\ \boldsymbol{b}_1;\ \ldots;\ \boldsymbol{b}_n;\end{matrix}\ x_1,\ldots,x_n\right)t^m$$
$$=\frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta}F^{p_0:\,p_1;\,\ldots;\,p_n}_{q_0:\,q_1;\,\ldots;\,q_n}\begin{pmatrix}-x_1\zeta\\ \vdots\\ -x_n\zeta\end{pmatrix}, \tag{29}$$

where α, β are arbitrary complex numbers, and ζ is a function of t defined (implicitly) by

$$\zeta=t(1+\zeta)^{\beta+1},\qquad \zeta(0)=0; \tag{30}$$

$$\sum_{m=0}^{\infty}\frac{(\lambda)_m}{m!}F^{1+p_0:\,p_1;\,\ldots;\,p_n}_{q_0:\,q_1;\,\ldots;\,q_n}\left(\begin{matrix}\lambda+m,\ \boldsymbol{a}_0:\ \boldsymbol{a}_1;\ \ldots;\ \boldsymbol{a}_n;\\ \boldsymbol{b}_0:\ \boldsymbol{b}_1;\ \ldots;\ \boldsymbol{b}_n;\end{matrix}\ x_1,\ldots,x_n\right)t^m$$
$$=(1-t)^{-\lambda}F^{1+p_0:\,p_1;\,\ldots;\,p_n}_{q_0:\,q_1;\,\ldots;\,q_n}\left(\begin{matrix}\lambda,\ \boldsymbol{a}_0:\ \boldsymbol{a}_1;\ \ldots;\ \boldsymbol{a}_n;\\ \boldsymbol{b}_0:\ \boldsymbol{b}_1;\ \ldots;\ \boldsymbol{b}_n;\end{matrix}\ \frac{x_1}{1-t},\ldots,\frac{x_n}{1-t}\right),\qquad |t|<1. \tag{31}$$

For $\beta=0$, (30) immediately yields

$$\zeta=t/(1-t),$$

so that (29) with $\alpha=\lambda-1$ and $\beta=0$ assumes the form (cf Srivastava and Choe 1972)

$$\sum_{m=0}^{\infty}\frac{(\lambda)_m}{m!}F^{1+p_0:\,p_1;\,\ldots;\,p_n}_{q_0:\,q_1;\,\ldots;\,q_n}\left(\begin{matrix}-m,\ \boldsymbol{a}_0:\ \boldsymbol{a}_1;\ \ldots;\ \boldsymbol{a}_n;\\ \boldsymbol{b}_0:\ \boldsymbol{b}_1;\ \ldots;\ \boldsymbol{b}_n;\end{matrix}\ x_1,\ldots,x_n\right)t^m$$
$$=(1-t)^{-\lambda}F^{1+p_0:\,p_1;\,\ldots;\,p_n}_{q_0:\,q_1;\,\ldots;\,q_n}\left(\begin{matrix}\lambda,\ \boldsymbol{a}_0:\ \boldsymbol{a}_1;\ \ldots;\ \boldsymbol{a}_n;\\ \boldsymbol{b}_0:\ \boldsymbol{b}_1;\ \ldots;\ \boldsymbol{b}_n;\end{matrix}\ -\frac{x_1t}{1-t},\ldots,-\frac{x_nt}{1-t}\right),\qquad |t|<1, \tag{32}$$

which is analogous to (31).

A limiting case of the generating function (32) follows in view of (26), and we obtain (cf Srivastava and Daoust 1973)

$$\sum_{m=0}^{\infty}F^{1+p_0:\,p_1;\,\ldots;\,p_n}\left(\begin{matrix}-m,\ a_0:\ a_1;\ldots;\ a_n;\\ b_0:\ b_1;\ldots;\ b_n;\end{matrix}\ x_1,\ldots,x_n\right)\frac{t^m}{m!}$$
$$=\mathrm{e}^tF^{p_0:\,p_1;\,\ldots;\,p_n}_{q_0:\,q_1;\,\ldots;\,q_n}\begin{pmatrix}-x_1t\\ \vdots\\ -x_nt\end{pmatrix}. \tag{33}$$

In a similar manner it is not difficult to deduce (from more general multiple-series identities) the following interesting generalisation of a generating function considered by Niukkanen (1983, p 1820, equation (33)):

$$F^{p_0:\, p_1;\, \dots;\, p_n}_{q_0:\, q_1;\, \dots;\, q_n}\begin{pmatrix} x_1 t \\ \vdots \\ x_n t \end{pmatrix} = \sum_{m=0}^{\infty} \frac{(\boldsymbol{a}_0)_m (\boldsymbol{a}_n)_m}{(\boldsymbol{b}_0)_m (\boldsymbol{b}_n)_m} \frac{x_n^m}{m!}$$

$$\times F^{1+q_n:\, p_1;\, \dots;\, p_{n-1}}_{p_n:\, q_1;\, \dots;\, q_{n-1}}\left({}^{-m,\; 1-m-\boldsymbol{b}_n:\; \boldsymbol{a}_1;\, \dots;\, \boldsymbol{a}_{n-1};}_{ 1-m-\boldsymbol{a}_n:\; \boldsymbol{b}_1;\, \dots;\, \boldsymbol{b}_{n-1};}\; (-1)^{\sigma} \frac{x_1}{x_n}, \dots, (-1)^{\sigma} \frac{x_{n-1}}{x_n} \right) t^m, \tag{34}$$

where, for convenience, $\sigma = 1 - p_n + q_n$. The special role of the variable x_n in (34) can be appropriately assumed instead by any of the remaining variables $x_1, \dots, x_{n-1}$.

In its special case when $p_0 = q_0 = 0$, this last generating function (34) reduces to Niukkanen's result referred to above.

Niukkanen (1983, 1984) has proved a number of operator reduction formulae for the multivariable hypergeometric function (1). All of Niukkanen's results of this type, including his main result (Niukkanen 1984, p L733, equation (20)), are obvious special cases of the following rather elementary theorem which can nonetheless be applied to derive a much larger set of operator reduction formulae involving various classes of multiple hypergeometric series.

Theorem. For bounded multiple sequences $\{\Lambda(s_1, \dots, s_n)\}$ and $\{\Omega(s_1, \dots, s_n)\}$, let

$$\mathscr{F}(x_1, \dots, x_n) = \sum_{s_1, \dots, s_n = 0}^{\infty} \Lambda(s_1, \dots, s_n) \frac{x_1^{s_1}}{s_1!} \cdots \frac{x_n^{s_n}}{s_n!} \tag{35}$$

and

$$\mathscr{G}(x_1, \dots, x_n) = \sum_{s_1, \dots, s_n = 0}^{\infty} \Omega(s_1, \dots, s_n) \frac{x_1^{s_1}}{s_1!} \cdots \frac{x_n^{s_n}}{s_n!}. \tag{36}$$

Also define

$$\begin{aligned} \mathscr{F} * \mathscr{G}(x_1, \dots, x_n) &= \sum_{s_1, \dots, s_n = 0}^{\infty} \Lambda(s_1, \dots, s_n) \Omega(s_1, \dots, s_n) \frac{x_1^{s_1}}{s_1!} \cdots \frac{x_n^{s_n}}{s_n!} \\ &= \mathscr{G} * \mathscr{F}(x_1, \dots, x_n). \end{aligned} \tag{37}$$

Then

$$\mathscr{F} * \mathscr{G}(x_1, \dots, x_n) = \mathscr{F}\left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}\right) \mathscr{G}(x_1 t_1, \dots, x_n t_n) \bigg|_{t_1 = \dots = t_n = 0}, \tag{38}$$

provided that each of the multiple series involved converges absolutely.

Proof. Since

$$\left(\frac{\partial}{\partial t_k}\right)^{s_k} (x_k t_k)^{m_k} = \frac{m_k!}{(m_k - s_k)!} x_k^{m_k} t_k^{m_k - s_k}, \qquad k = 1, \dots, n, \tag{39}$$

the second member of our assertion (38) equals

$$\sum_{s_1,\ldots,s_n=0}^{\infty} \frac{\Lambda(s_1,\ldots,s_n)}{s_1!\ldots s_n!} \sum_{m_1=s_1}^{\infty} \cdots \sum_{m_n=s_n}^{\infty} \Omega(m_1,\ldots,m_n) \prod_{k=1}^{n} \frac{x_k^{m_k} t_k^{m_k-s_k}}{(m_k-s_k)!}\bigg|_{t_1=\ldots=t_n=0}$$

$$= \sum_{s_1,\ldots,s_n=0}^{\infty} \frac{\Lambda(s_1,\ldots,s_n)}{s_1!\ldots s_n!} \sum_{m_1,\ldots,m_n=0}^{\infty} \Omega(m_1+s_1,\ldots,m_n+s_n)$$

$$\times \prod_{k=1}^{n} \frac{x_k^{m_k+s_k} t_k^{m_k}}{m_k!}\bigg|_{t_1=\ldots=t_n=0}$$

$$= \sum_{s_1,\ldots,s_n=0}^{\infty} \Lambda(s_1,\ldots,s_n)\Omega(s_1,\ldots,s_n)\frac{x_1^{s_1}}{s_1!}\cdots\frac{x_n^{s_n}}{s_n!},$$

which, in view of the definition (37), is precisely the first member of (38), and the proof of the theorem is evidently completed.

By assigning suitable special values to the essentially arbitrary coefficients $\Lambda(s_1,\ldots,s_n)$ and $\Omega(s_1,\ldots,s_n)$, our theorem can readily be applied to derive the aforementioned classes of operator reduction formulae for various hypergeometric functions of n variables. The details involved are fairly straightforward, and may be left as an exercise for the user of multivariable hypergeometric functions.

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